Universität Konstanz
Fachbereich Mathematik und Statistik
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Winter semester 2019/2020

Exercises for the course "Linear Algebra I"

Sheet 7

Hand in your solutions on Thursday, 12. Dezember 2019, 09:55, in the Postbox of your Tutor in F4. Please try to write your solutions clear and readable, write your name and the name of your tutor on each sheet, and staple the sheets together.

Exercise 7.1 ***Beweismechanikaufgabe*** *(2 points)*

Bitte gehen Sie in dieser Aufgabe nach den Regeln der Beweismechanik vor und geben Ihre Lösung auf einem separaten Blatt in den Briefkasten mit der Aufschrift „Beweismechanikaufgaben" (Briefkasten 18) ab. Ihnen unbekannte Begriffe und Symbole können Sie in der Beweismechanik nachschlagen.

Sei K ein Körper und V ein K-Vektorraum. Für $A, B \subset V$ definieren wir

$$A + B := \{x \in V : (\exists a \in A : \exists b \in B : x = a + b)\}.$$

Zeigen Sie: Sind $U_1, U_2, U_3 \subset V$ Unterräume von V mit $U_3 \subset U_1$, so gilt $(U_1 \cap U_2) + U_3 = U_1 \cap (U_2 + U_3)$.

Exercise 7.2 *(5 points)*

Let K be a field and let k be a subfield of K.

(a) Let $A, B \in M_{n\times m}(K)$ be in reduced row echelon form. Show that A is row equivalent to B if and only if $A = B$.
(b) Conclude that the reduced row echelon form of a matrix over a given field is uniquely determined.
(c) Let $A \in M_{n\times m}(k)$. Show that $rZSF_k(A) = rZSF_K(A)$ (i.e. that the reduced row echelon forms over K and k coincide).
(d) Let $A \in M_{n\times n}(k)$, and denote by S the homogenous linear equation $Ax = 0$. Show that S has a non-trivial solution in k^n if and only if S has a non-trivial solution in K^n.
(e) Let S be the same as in part (d). Which inclusions may occur between the solution sets $\mathcal{L}_k(S)$ and $\mathcal{L}_K(S)$? When precisely does which one apply?

Exercise 7.3 *(5 points)*

Let K be a field and let V be a K-vector space of dimension $\dim(V) = n$ for some $n \in \mathbb{N}$. A subset $X \subseteq V$ is said to be *minimal generating*, if $span(X) = V$ and $span(X \setminus \{x\}) \neq V$ for all $x \in X$. X is called *maximal linearly independent*, if X is linearly independent and $X \cup \{y\}$ is linearly dependent for all $y \in V \setminus X$. Prove that:

(a) If $X \subseteq V$ is a linearly independent set with n elements, then X is a basis of V.
(b) Let $X \subseteq V$ be finite such that $span(X) = V$. Show that there exists a basis Y of V such that $Y \subseteq X$. Does the conclusion also apply when X is an infinite set? Prove or disprove your answer.
(c) Let $X \subseteq V$ be a set with precisely n elements such that $span(X) = V$. Show that X is a basis of V.
(d) Let $X \subseteq V$ be arbitary. Show that the following statements are equivalent:
  (i) X is a basis of V.
  (ii) X is maximal linearly independent.
  (iii) X is minimal generating.

Exercise 7.4 *(4 points)*

(Please note reverse side!) Let K be a field and $A \in M_{n\times n}(K)$. A is called *symmetric*, if $A_{ij} = A_{ji}$ for all $1 \leq i,\ j \leq n$. If $\text{Char}(K) \neq 2$, A is called *alternating*, if $A_{ij} = -A_{ji}$ for all $1 \leq i,\ j \leq n$. We denote the symmetric $n \times n$-matrices by $Sym_{n\times n}(K)$, and the alternating $n \times n$-matrices by $Alt_{n\times n}(K)$.

Let K be a field with $\text{Char}(K) \neq 2$.

(a) Determine the dimension and a basis for both $Sym_{n\times n}(K)$ and $Alt_{n\times n}(K)$.
(b) Determine $Sym_{n\times n}(K) + Alt_{n\times n}(K)$ and $Sym_{n\times n}(K) \cap Alt_{n\times n}(K)$.

Now let K be a field of characteristic 2, and let $Alt_{n\times n}(K)$ be defined as above. Determine again the dimension of $Alt_{n\times n}(K)$, as well as $Sym_{n\times n}(K) + Alt_{n\times n}(K)$ and $Sym_{n\times n}(K) \cap Alt_{n\times n}(K)$.

Hint: You may without proof assume that $Sym_{n\times n}(K)$ and $Alt_{n\times n}(K)$ are subspaces of the K-vector space $M_{n\times n}(K)$.

Excercise 7.5

Come to the Christmas party of our department.